Covariant representation of microscopic charge and current densities in terms of polarisation and magnetisation fields

1978 J. Phys. A: Math. Gen. 11 1899

(http://iopscience.iop.org/0305-4470/11/10/009)

J. Phys. A: Math. Gen., Vol. 11, No. 10, 1978. Printed in Great Britain

Covariant representation of microscopic charge and current densities in terms of polarisation and magnetisation fields

W P Healy

Research School of Chemistry, The Australian National University, PO Box 4, Canberra, ACT 2600, Australia

Received 6 April 1978

Abstract. It is shown that a previously developed formalism for representing microscopic charge and current densities in terms of polarisation and magnetisation fields can be written in a manifestly Lorentz covariant manner. The charge–current density four-vector associated with an aggregate of charged point particles is first constructed and its behaviour under the time reversal transformation is discussed. Particular polarisation–magnetisation tensors defined as sums of line integrals of delta functions are then shown to reproduce the charge and current densities in the required fashion; it is noted that the components of these tensors, although involving instantaneous integrals along spatial curves, are defined in the same way by all observers related by homogeneous Lorentz transformations, provided only that the speed of any point of the curves is less than c. The general polarisation–magnetisation tensor is derived through a transformation generated by an arbitrary pseudovector field. The explicit form of this field, as well as of a subsidiary pseudoscalar field, is obtained for those transformations that interrelate polarisation–magnetisation ensors of the line integral kind.

1. Introduction

In a recent paper (Healy 1977, to be referred to as I) the microscopic charge and current densities due to an aggregate of charged point particles were expressed in terms of polarisation and magnetisation fields. It was shown that particular polarisation and magnetisation fields can be defined as sums of line integrals of delta functions along curves joining an arbitrarily moving reference point to the positions of the particles, and that all possible polarisation and magnetisation fields are generated through a transformation involving arbitrary differentiable scalar and vector fields. Covariant treatments of microscopic polarisation and magnetisation fields have been given previously by de Groot and coworkers (de Groot and Vlieger 1965, de Groot and Suttorp 1965, de Groot 1969). In the present paper we show that the general formalism developed in I can also be written in a manifestly Lorentz covariant manner, so that the agreement of this theory with the principles of special relativity is established. The behaviour of the fields under all homogeneous Lorentz transformations is investigated; in particular, the effect of the time reversal transformation, which was not considered in I, is discussed in some detail.

In the relativistic notation to be used the contravariant components of the space-time four-vector are given by

$$(x^{\mu}) = (x^0, x^i) = (ct, \boldsymbol{r}) \tag{1}$$

and the covariant components of the four-dimensional gradient by

$$(\partial_\mu)=(\partial_0,\partial_i)=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right). \tag{2}$$

The fundamental metric tensor g is chosen so that

$$g^{\mu\nu}=g_{\mu\nu}=\begin{cases}1, & \text{if } \mu=\nu=0\\ -1, & \text{if } \mu=\nu\neq 0,\\ 0, & \text{if } \mu\neq\nu\end{cases} \qquad \mu,\nu=0,1,2,3. \tag{3}$$

A repeated Greek index (apart from α, which will be used to label the particles in the aggregate) will denote a sum from 0 to 3 and a repeated Latin index a sum from 1 to 3. The Maxwell–Lorentz equations then take the form

$$\partial_\nu b^{\mu\nu}(x)=\frac{4\pi}{c}j^\mu(x) \tag{4a}$$

$$\partial_\nu b^{*\mu\nu}(x)=0 \tag{4b}$$

where j is the charge–current density four-vector,

$$(j^\mu)=(c\rho,\boldsymbol{j}), \tag{5}$$

and b the antisymmetric electromagnetic field tensor,

$$b^{0i}=e^i,\qquad b^{ij}=\epsilon^{ijk}b^k. \tag{6}$$

The tensor b^* is the dual of b and is defined by

$$b^{*\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}b_{\rho\sigma}. \tag{7}$$

Here the ϵ^{ijk} are the components of the three-dimensional alternating pseudotensor and the $\epsilon^{\mu\nu\rho\sigma}$ are those of its four-dimensional counterpart. The atomic field equations that involve sources have the covariant form

$$\partial_\nu h^{\mu\nu}(x)=\frac{4\pi}{c}j^\mu_{\text{true}}(x) \tag{8}$$

where j_{true} is the 'true' charge–current density four-vector and h is the antisymmetric electric displacement–magnetic field tensor with components given by

$$h^{0i}=d^i,\qquad h^{ij}=\epsilon^{ijk}h^k. \tag{9}$$

The true or atomic charge and current densities are those of a collective point particle coincident with the chosen reference point and having charge equal to the total charge of the aggregate. Introducing the antisymmetric polarisation–magnetisation tensor m through the equation

$$h^{\mu\nu}=b^{\mu\nu}-4\pi m^{\mu\nu} \tag{10}$$

we obtain from equations (4a) and (8) the fundamental decomposition (11) of the charge-current density four-vector into true and bound contributions,

$$j^\mu=j^\mu_{\text{true}}+c\partial_\nu m^{\mu\nu}. \tag{11}$$

The components of m are related to those of the polarisation and magnetisation fields by

$$m^{0i} = -p^i, \qquad m^{ij} = \epsilon^{ijk} m^k. \tag{12}$$

In § 3 below particular polarisation–magnetisation tensors defined as sums of line integrals of delta functions will be discussed; these tensors will be displayed in a manifestly covariant form and it will be verified that they satisfy equation (11). The general solution of equation (11) will be obtained in § 4. It will first be necessary, however, to show that the microscopic charge and current densities transform like a four-vector under homogeneous Lorentz transformations, and this will be done in § 2.

2. The charge–current density four-vector

The physical system to be considered consists of an aggregate of charged point particles in motion. An observer O in an inertial frame of reference ascribes to a particle labelled α the charge e_α, the position vector $\boldsymbol{q}_\alpha$ (with rectangular Cartesian components q^i_α) and the velocity vector $\dot{\boldsymbol{q}}_\alpha$, where the dot denotes differentiation with respect to the time t. Associated with particle α we thus have the four-vector x_α with components given by

$$(x^\mu_\alpha) = (ct, \boldsymbol{q}_\alpha). \tag{13}$$

Under a homogeneous Lorentz transformation Λ to another observer $\bar{\text{O}}$, these components transform like the space–time coordinates x^μ. To fulfil the conditions for a Lorentz transformation the elements $\Lambda^\mu{}_\nu$ must be real and satisfy

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} = g^{\mu\nu}. \tag{14}$$

This equation implies that the determinant $|\Lambda|$ of the matrix $(\Lambda^\mu{}_\nu)$ is either $+1$ (proper Lorentz transformation) or -1 (improper Lorentz transformation) and that $\Lambda^0{}_0$ is either greater than or equal to 1 (orthochronous Lorentz transformation) or less than or equal to -1 (non-orthochronous Lorentz transformation). A particular transformation which will be of importance in what follows is the time reversal transformation T,

$$\bar{t} = -t, \qquad \bar{x}^i = x^i, \tag{15}$$

for which $|\Lambda| = -1$ and $\Lambda^0{}_0 = -1$. In addition to this we shall consider the space inversion or parity transformation P,

$$\bar{t} = t, \qquad \bar{x}^i = -x^i, \tag{16}$$

for which $|\Lambda| = -1$ and $\Lambda^0{}_0 = 1$, and the charge conjugation transformation C which relates two observers O and $\bar{\text{O}}$ employing the same space–time coordinate frame but adopting different conventions for the signs of the charges.

We now construct the charge–current density four-vector that corresponds to the aggregate of particles. To do this it is necessary to express the time in any frame as a function of an invariant parameter τ (cf de Groot 1969). We can find such a parameter in the following way. Let us first choose a point P that traces out a path with speed, in any inertial frame, less than c. Subject to this restriction the motion of P can be arbitrary, so long as it is also sufficiently smooth, and need not be related to the motion of any material particle. We take τ to be the proper time associated with the motion of P, with the added condition that the instantaneous rest frame of P is always chosen so that time increases in it, and thus so that τ increases continuously from $-\infty$ to ∞ as P describes its path. The differential $d\tau$ is then given by the formula

$$d\tau = \pm dt\sqrt{1-V^2/c^2} \tag{17}$$

where $V(t)$ is the velocity of P as measured by an observer O at his time t and where the plus sign is to be used if the Lorentz transformation connecting O's frame and the instantaneous rest frame is orthochronous ($dt/d\tau>0$) and the minus sign is to be used if the transformation is non-orthochronous ($dt/d\tau<0$). The τ defined in this manner is determined up to an arbitrary constant and is invariant under all Lorentz transformations. Because the particle coordinates q^i_α are functions of t, the components x^μ_α of the four-vectors x_α can all be expressed as definite functions $x^\mu_\alpha(\tau)$ of the parameter τ. It follows that the quantities $j^\mu(x)$ defined by

$$j^\mu(x) = c\sum_\alpha e_\alpha \int_{-\infty}^{\infty} \frac{dx^\mu_\alpha}{d\tau'}\delta\{x - x_\alpha(\tau')\}\, d\tau' \tag{18}$$

are the components of a four-vector, since the four-dimensional delta function is an invariant function. Now if the time t' corresponds to τ' just as t corresponds to τ, then

$$\delta(ct-ct') = \frac{1}{c|dt'/d\tau'|}\delta(\tau-\tau'). \tag{19}$$

This holds because $dt'/d\tau'$ is never zero and because for fixed t the function $t'(\tau')-t$ has only one zero, namely τ. Using the relation (19) to do the τ' integration in equation (18) we obtain

$$j^0(x) = \pm\sum_\alpha e_\alpha c\,\delta\{\boldsymbol{r} - \boldsymbol{q}_\alpha(t)\} \tag{20a}$$

$$j^i(x) = \pm\sum_\alpha e_\alpha \dot{q}^i_\alpha(t)\,\delta\{\boldsymbol{r} - \boldsymbol{q}_\alpha(t)\} \tag{20b}$$

where the plus or the minus sign is applicable according as $dt/d\tau$ is greater or less than zero. The time and space parts of the four-vector j are thus essentially the charge and current densities due to the aggregate of particles.

The appearance of the minus signs in equations (20) requires some comment. These signs occur when $dt/d\tau$ is less than zero, that is whenever the minus sign appears in equation (17), and they ensure that j transforms as a true four-vector under all, including non-orthochronous, homogeneous Lorentz transformations. Similarly the minus sign in equation (17) ensures that $d\tau$ is a true invariant. It is possible to choose the parameter τ in such a way that only the plus sign occurs in equation (17); $d\tau$ would then be the same for all observers related by orthochronous transformations but would suffer a change of sign under a non-orthochronous transformation. In addition, the minus signs would never occur in equations (20). However, j would no longer be a true four-vector, as its components would transform according to the rule

$$\bar{j}^\mu(\bar{x}) = \mathrm{sgn}(\Lambda^0{}_0)\Lambda^\mu{}_\nu j^\nu(x) \tag{21}$$

where sgn is the signum function. We shall adhere to the convention adopted above, namely that $d\tau$ is a true invariant and j a true four-vector, so that the charge and current densities are as given in equations (20). This convention then determines the transformation behaviour of the other vector and tensor fields; b, m and h are true tensors while b^* is a pseudotensor with components satisfying the transformation equations

$$\bar{b}^{*\mu\nu}(\bar{x}) = |\Lambda|\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}b^{*\rho\sigma}(x). \tag{22}$$

Under the charge conjugation transformation C all of these fields change sign. The power–force density four-vector f, on the other hand, remains invariant under C. The components of f, which is a true four-vector, are given by

$$f^{\mu} = -\frac{1}{c}b^{\mu\nu}j_{\nu}, \tag{23}$$

so that

$$(f^{\mu}) = \left(\frac{1}{c}w, \boldsymbol{f}\right) \tag{24}$$

where w is the power density $\boldsymbol{j}\,.\,\boldsymbol{e}$ and $\boldsymbol{f}$ is the Lorentz force density $\rho\boldsymbol{e} + (1/c)\boldsymbol{j}\times\boldsymbol{b}$. Invariance under C is of course shared by f with forces that are not of electromagnetic origin. Also the behaviour of f under non-orthochronous transformations is independent of the convention used for the behaviour of j and hence of b; the charges and fields can be observed only in each other's presence and their combination on the right hand side of equation (23) is such as to ensure transformation properties for f that would be expected for ordinary forces. To illustrate the consequences of the sign convention used in equations (17) and (20), the transformation properties of the fields under T, as well as under P and C, are shown in table 1. It may be noted from the effect of the P transformation that $\boldsymbol{j}$, $\boldsymbol{e}$, $\boldsymbol{p}$, $\boldsymbol{d}$ and $\boldsymbol{f}$ are three-dimensional polar vectors, whereas $\boldsymbol{b}$, $\boldsymbol{m}$ and $\boldsymbol{h}$ are three-dimensional axial vectors.

The true charge and current densities can be expressed in a covariant form analogous to that of the total charge and current densities. We choose for the

Table 1. Field transformation properties under T, P and C. Components of a four-vector or tensor referred to two reference frames related by T, P or C are distinguished by the pairs of suffixes ($\rightarrow$, $\leftarrow$), (L, R) or (+, −), respectively. The fields are all evaluated at the same world point; thus, for example, $\rho_{\rightarrow} = -\rho_{\leftarrow}$ means that $\rho_{\rightarrow}(x_{\rightarrow}) = -\rho_{\leftarrow}(x_{\leftarrow})$, where $x^{\mu}_{\rightarrow}$ and $x^{\mu}_{\leftarrow}$ are the space–time coordinates ascribed to the same physical event by two observers related by T.

	T	P	C
j	$\rho_{\rightarrow} = -\rho_{\leftarrow}$ $j^i_{\rightarrow} = +j^i_{\leftarrow}$	$\rho_{\mathrm{L}} = +\rho_{\mathrm{R}}$ $j^i_{\mathrm{L}} = -j^i_{\mathrm{R}}$	$\rho_+ = -\rho_-$ $j^i_+ = -j^i_-$
m	$p^i_{\rightarrow} = -p^i_{\leftarrow}$ $m^i_{\rightarrow} = +m^i_{\leftarrow}$	$p^i_{\mathrm{L}} = -p^i_{\mathrm{R}}$ $m^i_{\mathrm{L}} = +m^i_{\mathrm{R}}$	$p^i_+ = -p^i_-$ $m^i_+ = -m^i_-$
b	$e^i_{\rightarrow} = -e^i_{\leftarrow}$ $b^i_{\rightarrow} = +b^i_{\leftarrow}$	$e^i_{\mathrm{L}} = -e^i_{\mathrm{R}}$ $b^i_{\mathrm{L}} = +b^i_{\mathrm{R}}$	$e^i_+ = -e^i_-$ $b^i_+ = -b^i_-$
h	$d^i_{\rightarrow} = -d^i_{\leftarrow}$ $h^i_{\rightarrow} = +h^i_{\leftarrow}$	$d^i_{\mathrm{L}} = -d^i_{\mathrm{R}}$ $h^i_{\mathrm{L}} = +h^i_{\mathrm{R}}$	$d^i_+ = -d^i_-$ $h^i_+ = -h^i_-$
f	$w_{\rightarrow} = -w_{\leftarrow}$ $f^i_{\rightarrow} = +f^i_{\leftarrow}$	$w_{\mathrm{L}} = +w_{\mathrm{R}}$ $f^i_{\mathrm{L}} = -f^i_{\mathrm{R}}$	$w_+ = +w_-$ $f^i_+ = +f^i_-$

aggregate a reference point $\boldsymbol{R}(t)$, which may move arbitrarily subject only to the condition that $|\dot{\boldsymbol{R}}| < c$, and from it construct the four-vector X with components

$$(X^{\mu}) = (ct, \boldsymbol{R}). \tag{25}$$

These components can be expressed as functions of the parameter τ, although the motion of the reference point need not be related to that of the point P introduced earlier. The true charge–current density four-vector is then given by

$$j^{\mu}_{\text{true}}(x) = cQ \int_{-\infty}^{\infty} \frac{\mathrm{d}X^{\mu}}{\mathrm{d}\tau'} \delta\{x - X(\tau')\} \, \mathrm{d}\tau' \tag{26}$$

where Q is the total charge of the aggregate.

3. Line integral form of polarisation–magnetisation tensor

3.1. Construction of line integral polarisation–magnetisation tensor

In this section we show that there are line integral polarisation–magnetisation tensors satisfying equation (11). The components of such tensors are related through equations (12) to line integral polarisation and magnetisation fields defined by

$$\boldsymbol{p}(\boldsymbol{r}, t) = \pm \sum_{\alpha} e_{\alpha} \int_{C_{\alpha}} \mathrm{d}\boldsymbol{r}' \, \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{27a}$$

and

$$\boldsymbol{m}(\boldsymbol{r}, t) = \pm \sum_{\alpha} \frac{e_{\alpha}}{c} \int_{C_{\alpha}} \mathrm{d}\boldsymbol{r}' \times \dot{\boldsymbol{r}}' \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{27b}$$

Here C_{α} is an arbitrary smoothly moving curve connecting the reference point $\boldsymbol{R}$ to the position of particle α and the plus or minus sign is to be used according as $\mathrm{d}t/\mathrm{d}\tau$ is greater or less than zero. The points of C_{α} can be specified by a real parameter u_{α} varying between the lower limit $u_{\alpha}^{(1)}$ corresponding to the reference point and the upper limit $u_{\alpha}^{(2)}$ corresponding to the position of particle α. These limits will be supposed to be independent of the time, although they may change from curve to curve. We shall also assume that the curves are parametrised in the same way by every observer, so that a particular value of u_{α} corresponds to a unique and objectively identifiable point of C_{α}. It follows that the u_{α} are Lorentz invariant parameters. An observer O describes the C_{α} by functions $\boldsymbol{r}'(u_{\alpha}, t)$ which specify the configuration of each curve at time t. These functions are the spatial parts of the four-vector functions given by

$$\{x'^{\mu}(u_{\alpha}, \tau)\} = \{ct(\tau), \boldsymbol{r}'(u_{\alpha}, t(\tau))\}. \tag{28}$$

It should be noted that $\partial x'^{0}/\partial u_{\alpha}$ is identically zero; this is so because for a given τ the events with coordinates $x'^{\mu}(u_{\alpha}, \tau)$ refer to the whole curve C_{α} at a fixed time t and are therefore simultaneous events for O. Let us consider the antisymmetric tensor m defined by

$$m^{\mu\nu}(x) = \sum_{\alpha} e_{\alpha} \int_{-\infty}^{\infty} \int_{u_{\alpha}^{(1)}}^{u_{\alpha}^{(2)}} \left[\frac{\partial x'^{\mu}}{\partial u_{\alpha}} \frac{\partial x'^{\nu}}{\partial \tau'} - \frac{\partial x'^{\nu}}{\partial u_{\alpha}} \frac{\partial x'^{\mu}}{\partial \tau'} \right] \delta(x - x') \, \mathrm{d}u_{\alpha} \, \mathrm{d}\tau'. \tag{29}$$

By using the relation (19) to do the τ' integration it may be verified that the components of m are formed, as in equations (12), from the components of the polarisation and magnetisation fields defined by equations (27). To show that m^{i0} is equal to p^i it is necessary also to use the fact that $\partial x'^0/\partial u_\alpha$ is identically zero. Now from equation (29) we obtain, after integrating by parts with respect to τ',

$$\partial_\nu m^{\mu\nu} = -\sum_\alpha e_\alpha \int_{u_\alpha^{(1)}}^{u_\alpha^{(2)}} \left\{ \left[\frac{\partial x'^\mu}{\partial u_\alpha} \delta(x-x') \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial u_\alpha} \left[\frac{\partial x'^\mu}{\partial \tau'} \delta(x-x') \right] \mathrm{d}\tau' \right\} \mathrm{d}u_\alpha. \tag{30}$$

The boundary terms vanish because $\delta(ct-ct')$ is zero when τ' is $\pm\infty$ and τ is finite. The u_α integration can then be done to give

$$c\partial_\nu m^{\mu\nu} = c\sum_\alpha e_\alpha \int_{-\infty}^{\infty} \frac{\mathrm{d}x_\alpha^\mu}{\mathrm{d}\tau'} \delta(x-x_\alpha)\, \mathrm{d}\tau' - cQ \int_{-\infty}^{\infty} \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau'} \delta(x-X)\, \mathrm{d}\tau' = j^\mu - j^\mu_{\mathrm{true}}. \tag{31}$$

This proves that the line integral polarisation–magnetisation tensor affords a particular solution to equation (11).

3.2. Lorentz transformation of line integral polarisation–magnetisation tensor

The tensor m in equation (29) was defined through its components referred to an observer O. Under a Lorentz transformation Λ to another observer $\bar{\mathrm{O}}$ these components are transformed to

$$\bar{m}^{\mu\nu}(\bar{x}) = \sum_\alpha e_\alpha \int_{-\infty}^{\infty} \int_{u_\alpha^{(1)}}^{u_\alpha^{(2)}} \left\{ \frac{\partial \bar{x}'^\mu}{\partial u_\alpha} \frac{\partial \bar{x}'^\nu}{\partial \tau'} - \frac{\partial \bar{x}'^\nu}{\partial u_\alpha} \frac{\partial \bar{x}'^\mu}{\partial \tau'} \right\} \delta(\bar{x}-\bar{x}')\, \mathrm{d}u_\alpha\, \mathrm{d}\tau' \tag{32}$$

where

$$\bar{x}'^\mu(u_\alpha, \tau) = \Lambda^\mu{}_\nu x'^\nu(u_\alpha, \tau). \tag{33}$$

We have already noted that $\partial x'^0/\partial u_\alpha$ is identically zero and that because of this the polarisation field $\boldsymbol{p}$ associated with the components $m^{\mu\nu}$ is of the form (27*a*). It follows from equation (33), however, that $\partial \bar{x}'^0/\partial u_\alpha$ is in general non-zero. We shall show that despite this the components $\bar{m}^{\mu\nu}$ can, by a suitable transformation, be related to fields $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{m}}$ similar in form to the fields $\boldsymbol{p}$ and $\boldsymbol{m}$ given by equations (27).

For fixed τ' the events labelled $x'(u_\alpha, \tau')$ by O correspond to the curve C_α at the time t'. These same events are labelled $\bar{x}'(u_\alpha, \tau')$ by $\bar{\mathrm{O}}$ but they do not correspond to the configuration of C_α, as seen by $\bar{\mathrm{O}}$, at any one time $\bar{t}'$; instead, these events correspond to the different points $\bar{\boldsymbol{r}}'$ of C_α seen by $\bar{\mathrm{O}}$ at different times $(1/c)\bar{x}'^0(u_\alpha, \tau')$. That these times vary with u_α as well as with τ' is a reflection of the relativity of simultaneity—spatially separated events that are simultaneous for O are not, in general, simultaneous for $\bar{\mathrm{O}}$. Now if the speed of any point of C_α is always less than c, then $\partial \bar{x}'^0/\partial \tau'$ has the same sign for all u_α as well as for all τ'. We can prove this by noting that

$$\frac{\partial \bar{x}'^0}{\partial \tau'} = \left(\Lambda^0{}_0 + \Lambda^0{}_i \frac{1}{c} \frac{\partial r'^i}{\partial t'} \right) \frac{\mathrm{d}x'^0}{\mathrm{d}\tau'} \tag{34}$$

and that, because of the Cauchy–Schwartz inequality and equation (14),

$$\left| \Lambda^0{}_i \frac{1}{c} \frac{\partial r'^i}{\partial t'} \right|^2 \leqslant \left\{ \sum_{i=1}^{3} (\Lambda^0{}_i)^2 \right\} \frac{1}{c^2} \left(\frac{\partial \boldsymbol{r}'}{\partial t'} \right)^2 \leqslant \sum_{i=1}^{3} (\Lambda^0{}_i)^2 < (\Lambda^0{}_0)^2. \tag{35}$$

Thus $\Lambda^0{}_0 + \Lambda^0{}_i(c^{-1}\partial r'^i/\partial t')$ has the same sign as $\Lambda^0{}_0$ and so $\partial \bar{x}'^0/\partial\tau'$ has the same sign as or the opposite sign to $dx'^0/d\tau'$ according as $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. It follows that the sign of $\partial\bar{x}'^0/\partial\tau'$ is independent of u_α; moreover, since the sign of $dx'^0/d\tau'$ is independent of τ', the sign of $\partial\bar{x}'^0/\partial\tau'$ is independent of τ' as well. Let us again fix a proper time τ' for the moving point P. This determines, as was explained in § 2, a time $\bar{t}'(\tau')$ for observer $\bar{\mathrm{O}}$ which is of necessity independent of u_α. We let $\bar{\tau}'$ be the proper time at which $\bar{x}'^0$ equals $c\bar{t}'$. $\bar{\tau}'$ depends on $\bar{t}'$, and hence on τ', and also on u_α, and is such that

$$\bar{x}'^0\{u_\alpha, \bar{\tau}'(u_\alpha, \tau')\} = c\bar{t}'(\tau'). \tag{36}$$

The events labelled $\bar{x}'\{u_\alpha, \bar{\tau}'(u_\alpha, \tau')\}$ are thus all simultaneous for $\bar{\mathrm{O}}$—they correspond to the curve C_α as seen by $\bar{\mathrm{O}}$ at time $\bar{t}'(\tau')$. If these events are relabelled through the transformation

$$\bar{y}'(u_\alpha, \tau') \equiv \bar{x}'\{u_\alpha, \bar{\tau}'(u_\alpha, \tau')\}, \tag{37}$$

then it follows from equation (36) that $\partial\bar{y}'^0/\partial u_\alpha$ is identically zero. We now show that $\bar{m}^{\mu\nu}$ as given in equation (32) can be written as

$$\bar{m}^{\mu\nu}(\bar{x}) = \sum_\alpha e_\alpha \int_{-\infty}^{\infty}\int_{u_\alpha^{(1)}}^{u_\alpha^{(2)}} \left\{\frac{\partial\bar{y}'^\mu}{\partial u_\alpha}\frac{\partial\bar{y}'^\nu}{\partial\tau'} - \frac{\partial\bar{y}'^\nu}{\partial u_\alpha}\frac{\partial\bar{y}'^\mu}{\partial\tau'}\right\}\delta(\bar{x}-\bar{y}')\,du_\alpha\,d\tau', \tag{38}$$

so that, since $\partial\bar{y}'^0/\partial u_\alpha$ is zero, $\bar{m}^{\mu\nu}(\bar{x})$ bears the same relation to the polarisation and magnetisation fields $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{m}}$ defined by $\bar{\mathrm{O}}$ as instantaneous line integrals along the curves C_α that $m^{\mu\nu}(x)$ does to the fields $\boldsymbol{p}$ and $\boldsymbol{m}$ defined in a similar way by O. For the delta function that appears in equation (32) we have the decomposition

$$\delta\{\bar{x} - \bar{x}'(u_\alpha, \tau')\} = \pm\delta\{\bar{\boldsymbol{r}} - \bar{\boldsymbol{r}}'(u_\alpha, t'(\tau'))\}\frac{1}{\partial\bar{x}'^0/\partial\tau'}\delta\{\tau' - \bar{\tau}(u_\alpha, \tau)\} \tag{39}$$

where the plus or the minus sign is to be used according as $\partial\bar{x}'^0/\partial\tau'$ is greater or less than zero. Equation (39) holds because $\bar{\tau}(u_\alpha, \tau)$ is for fixed $\bar{t}$ the unique zero of $\bar{x}'^0(u_\alpha, \tau') - \bar{x}^0$ and because $\partial\bar{x}'^0/\partial\tau'$ is never zero. Substituting from equation (39) in equation (32) and using the relation (37) we obtain

$$\bar{m}^{\mu\nu}(\bar{x}) = \pm\sum_\alpha e_\alpha \int_{u_\alpha^{(1)}}^{u_\alpha^{(2)}} \left\{\frac{\partial\bar{y}'^\mu}{\partial u_\alpha}\frac{\partial\bar{y}'^\nu}{\partial\tau} - \frac{\partial\bar{y}'^\nu}{\partial u_\alpha}\frac{\partial\bar{y}'^\mu}{\partial\tau}\right\}\delta\{\bar{\boldsymbol{r}} - \bar{\boldsymbol{s}}'(u_\alpha, \bar{t}(\tau))\}\frac{1}{d\bar{y}'^0/d\tau}\,du_\alpha. \tag{40}$$

Here

$$\bar{\boldsymbol{s}}'(u_\alpha, \bar{t}(\tau)) = \bar{\boldsymbol{r}}'\{u_\alpha, t'(\bar{\tau}(u_\alpha, \tau))\} \tag{41}$$

and is for $\bar{\mathrm{O}}$ the position vector at time $\bar{t}$ of that point of C_α which corresponds to u_α. The criterion for the use of the plus or the minus sign in equation (40) is such as to ensure that $\pm d\bar{y}'^0/d\tau$ is positive. The right hand side of equation (38) therefore reduces to that of equation (40), and the equivalence of the expressions (32) and (38) for $\bar{m}^{\mu\nu}$ is established.

4. General form of polarisation–magnetisation tensor

4.1. Transformation of tensor field m with change of reference point $\boldsymbol{R}$

The line integral polarisation–magnetisation tensor defined by equation (29) is not the most general solution of equation (11). To find the general solution we consider two tensor fields $m_{(1)}$ and $m_{(2)}$ that are associated with the reference points $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ and that satisfy equation (11). These fields are not necessarily of the line integral form and the reference points are not necessarily coincident. We let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be joined by a curve C_{12} which degenerates to a point if the reference points coincide but which otherwise is specified by a Lorentz invariant parameter v that takes values between the limits $v^{(1)}$ corresponding to $\boldsymbol{R}_1$ and $v^{(2)}$ corresponding to $\boldsymbol{R}_2$. An observer O describes the points of C_{12} at a given time by a set of four-vectors with components $x'^{\mu}(v,\tau)$ which are such that $\partial x'^0/\partial v$ is identically zero. Defining the tensor $m_{(12)}$ by

$$m^{\mu\nu}_{(12)}(x)=Q\int_{-\infty}^{\infty}\int_{v^{(1)}}^{v^{(2)}}\left\{\frac{\partial x'^{\mu}}{\partial v}\frac{\partial x'^{\nu}}{\partial \tau'}-\frac{\partial x'^{\nu}}{\partial v}\frac{\partial x'^{\mu}}{\partial \tau'}\right\}\delta(x-x')\,\mathrm{d}v\,\mathrm{d}\tau' \tag{42}$$

we obtain

$$\begin{aligned}c\,\partial_\nu m^{\mu\nu}_{(12)}(x)&=cQ\int_{-\infty}^{\infty}\left[\frac{\partial x'^{\mu}}{\partial \tau'}\delta(x-x')\right]_{v^{(1)}}^{v^{(2)}}\mathrm{d}\tau'\\&=j^{\mu}_{\mathrm{true}(2)}(x)-j^{\mu}_{\mathrm{true}(1)}(x)\\&=c\,\partial_\nu\{m^{\mu\nu}_{(1)}(x)-m^{\mu\nu}_{(2)}(x)\}.\end{aligned} \tag{43}$$

It follows that

$$m^{\mu\nu}_{(2)}(x)=m^{\mu\nu}_{(1)}(x)-m^{\mu\nu}_{(12)}(x)+\epsilon^{\mu\nu\rho\sigma}\partial_\rho v_{(12)\sigma}(x) \tag{44}$$

where $v_{(12)}$ is a pseudo four-vector field. This is so because the necessary and sufficient condition for the four-divergence of an antisymmetric tensor to vanish identically is that the tensor be expressible in the form $\epsilon^{\mu\nu\rho\sigma}\partial_\rho v_\sigma$ for some v. (This in turn is equivalent to the three-dimensional theorems that the curl of a vector field vanishes identically if and only if the vector field is the gradient of a scalar potential and that the divergence of a vector field vanishes identically if and only if the vector field is the curl of a vector potential.) Equation (44) expresses the necessary and sufficient condition for both $m_{(1)}$ and $m_{(2)}$ to be solutions of equation (11) when either of them is so. The general solution of equation (11) is therefore obtained by starting with a particular solution—for example the line integral polarisation–magnetisation tensor discussed in § 3—and then performing the transformation (44) for all possible fields $v_{(12)}$ and all possible curves C_{12}.

If $m_{(1)}$ and $m_{(2)}$ are both of the line integral form (29) and are associated with curves $C_{\alpha 1}$ and $C_{\alpha 2}$ respectively, then the components of the field $v_{(12)}$ can be identified with linear combinations of integrals over surfaces Σ_α bounded by $C_{\alpha 1}$, $C_{\alpha 2}$ and C_{12}. The boundary curve of Σ_α will be denoted by Γ_α and will be taken to be traced out in the positive sense by going from $\boldsymbol{R}_1$ to $\boldsymbol{R}_2$ along C_{12}, then from $\boldsymbol{R}_2$ to $\boldsymbol{q}_\alpha$ along $C_{\alpha 2}$ and finally from $\boldsymbol{q}_\alpha$ back to $\boldsymbol{R}_1$ in the negative sense along $C_{\alpha 1}$. The convention used in this paper for the relation of the sense of a surface to that of its boundary curve (or of the sign of a volume to the sense of its boundary surface) is explained in the appendix. If we define $t^{\mu\nu}$ by

$$t^{\mu\nu}(x)=m^{\mu\nu}_{(2)}(x)-m^{\mu\nu}_{(1)}(x)+m^{\mu\nu}_{(12)}(x), \tag{45}$$

then we obtain from theorem 1 of the appendix

$$t^{i0}=\pm\sum_\alpha e_\alpha\oint_{\Gamma_\alpha}\mathrm{d}\boldsymbol{r}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')|^{i}=-\epsilon^{ijk}\partial_j v_{(12)k} \tag{46}$$

where

$$v_{(12)} = \pm \sum_\alpha e_\alpha \iint_{\Sigma_\alpha} \mathrm{d}\boldsymbol{s}'\, \delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{47}$$

and from theorems 2 and 4 we obtain

$$t^{ij} = \pm \epsilon^{ijk} \sum_\alpha \frac{e_\alpha}{c} \oint_{\Gamma_\alpha} \mathrm{d}\boldsymbol{r}' \times \dot{\boldsymbol{r}}'\, \delta(\boldsymbol{r}-\boldsymbol{r}')|^k = \epsilon^{ijk}[\partial_0 v_{(12)k} - \partial_k v_{(12)0}] \tag{48}$$

where

$$v^0_{(12)} = \pm \sum_\alpha \frac{e_\alpha}{c} \iint_{\Sigma_\alpha} \mathrm{d}\boldsymbol{s}' \cdot \dot{\boldsymbol{r}}'\, \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{49}$$

In these equations the choice between alternative signs is to be made so that the plus sign is used if $\mathrm{d}t/\mathrm{d}\tau > 0$ and the minus sign is used if $\mathrm{d}t/\mathrm{d}\tau < 0$. Since the $t^{\mu\nu}$ are the components of an antisymmetric tensor, equations (46) and (48) together are equivalent to

$$t^{\mu\nu}(x) = \epsilon^{\mu\nu\rho\sigma} \partial_\rho v_{(12)\sigma}(x). \tag{50}$$

Thus equation (44) holds for line integral polarisation and magnetisation fields and the components of $v_{(12)}$ are given for such fields by equations (47) and (49). We now introduce for the surface Σ_α parameters u_α and v_α such that on Σ_α

$$\mathrm{d}\boldsymbol{s}' = -\frac{\partial \boldsymbol{r}'}{\partial u_\alpha} \times \frac{\partial \boldsymbol{r}'}{\partial v_\alpha}\, \mathrm{d}u_\alpha\, \mathrm{d}v_\alpha. \tag{51}$$

An observer O assigns to the points of Σ_α the four-vector components $x'^\mu(u_\alpha, v_\alpha, \tau)$ and these, since they all refer to a given time for O, have the property that $\partial x'^0/\partial u_\alpha$ and $\partial x'^0/\partial v_\alpha$ are identically zero. It follows from this that equations (47) and (49) can be combined to give

$$v^\mu_{(12)}(x) = \epsilon^{\mu\nu\rho\sigma} \sum_\alpha e_\alpha \int_{-\infty}^{\infty} \int_{u^{(1)}_\alpha}^{u^{(2)}_\alpha} \int_{v^{(1)}_\alpha}^{v^{(2)}_\alpha} \frac{\partial x'_\nu}{\partial u_\alpha} \frac{\partial x'_\rho}{\partial v_\alpha} \frac{\partial x'_\sigma}{\partial \tau'} \delta(x-x')\, \mathrm{d}u_\alpha\, \mathrm{d}v_\alpha\, \mathrm{d}\tau', \tag{52}$$

which displays $v_{(12)}$ in a manifestly covariant form.

4.2. *Transformation of pseudovector field $v_{(12)}$ with change of curve C_{12}*

The curve C_{12} and the pseudovector field $v_{(12)}$ that specify the transformation (44) are not uniquely determined. Let us suppose that with two curves C^a_{12} and C^b_{12} joining $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are associated fields $m^{(a)}_{(12)}$ and $m^{(b)}_{(12)}$ defined as in equation (42), and that the fields $v^{(a)}_{(12)}$ and $v^{(b)}_{(12)}$ are such that in each case (a or b) the relation (44) holds between $m_{(1)}$ and $m_{(2)}$. We let S^{ab} be a surface bounded by and moving with the curves C^a and C^b (the suffix 12 now being suppressed); the boundary curve Γ^{ab} of S^{ab} consists of C^a taken in its positive sense and C^b in its negative sense and parameters v and w are chosen for the surface so that

$$\mathrm{d}\boldsymbol{s}' = -\frac{\partial \boldsymbol{r}'}{\partial w} \times \frac{\partial \boldsymbol{r}'}{\partial v}\, \mathrm{d}w\, \mathrm{d}v. \tag{53}$$

It follows from an analysis similar to the one leading to equation (50) that

$$\epsilon^{\mu\nu\rho\sigma} \partial_\rho \{v^{(b)}_\sigma(x) - v^{(a)}_\sigma(x)\} = -\{m^{(a)\mu\nu}(x) - m^{(b)\mu\nu}(x)\} = -\epsilon^{\mu\nu\rho\sigma} \partial_\rho v^{(ab)}_\sigma(x) \tag{54}$$

where

$$v^{(ab)\mu}(x)=\epsilon^{\mu\nu\rho\sigma}Q\int_{-\infty}^{\infty}\int_{w^{(1)}}^{w^{(2)}}\int_{v^{(1)}}^{v^{(2)}}\frac{\partial x'_\nu}{\partial w}\frac{\partial x'_\rho}{\partial v}\frac{\partial x'_\sigma}{\partial \tau'}\delta(x-x')\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}\tau'. \tag{55}$$

It can readily be shown that a sufficient condition for equation (54) to be valid is that there exists a pseudoscalar field $\bar{\chi}^{(ab)}$ such that

$$\begin{aligned}a^\mu(x)&\equiv v^{(b)\mu}(x)-v^{(a)\mu}(x)+v^{(ab)\mu}(x)\\&=\partial^\mu\bar{\chi}^{(ab)}(x).\end{aligned} \tag{56}$$

This condition is also necessary, for if $\epsilon^{\mu\nu\rho\sigma}\partial_\rho a_\sigma$ is identically zero, then taking μ, ν to be 0, i we see that $\epsilon^{ijk}\partial_j a_k$ is identically zero, and this implies that there exists a function $\chi^{(ab)}$ such that a^i is equal to $\partial^i\chi^{(ab)}$. Using this and taking μ, ν to be i, j we can verify that a^0 is of the form $\partial^0\chi^{(ab)}(x)+\psi^{(ab)}(x^0)$ for some function $\psi^{(ab)}$. We now define $\bar{\chi}^{(ab)}$ in a particular reference frame by

$$\bar{\chi}^{(ab)}(x)=\chi^{(ab)}(x)+\int^{x0}\psi^{(ab)}(x^{0'})\,\mathrm{d}x^{0'} \tag{57}$$

and extend the definition to every other frame in such a way that $\bar{\chi}^{(ab)}$ is a pseudoscalar function. Then equation (56) holds in the particular frame, by construction, and also in every other frame, by the covariant character of the equation. Thus the necessary and sufficient condition for the field $v^{(b)}$ associated with the curve C^b to generate from $m_{(1)}$ the same $m_{(2)}$ that the field $v^{(a)}$ associated with the curve C^a does is that

$$v^{(b)\mu}(x)=v^{(a)\mu}(x)-v^{(ab)\mu}(x)+\partial^\mu\bar{\chi}^{(ab)}(x) \tag{58}$$

where $v^{(ab)}$ is given by equation (55) and $\bar{\chi}^{(ab)}$ is a pseudoscalar function.

If $v^{(a)}$ and $v^{(b)}$ are defined, as in equation (52), to be linear combinations of integrals over surfaces Σ^a_α and Σ^b_α respectively, then an explicit form for $\bar{\chi}^{(ab)}$ can be found. We let V_α be the volume bounded by Σ^a_α, Σ^b_α and S^{ab} and assume that the total boundary surface σ_α is simple. (Otherwise a slightly more complicated formulation must be given.) We then have two cases according as the normal to S^{ab} has (i) the opposite sense to or (ii) the same sense as that to σ_α. From the definitions (52) and (55) and theorems 3 and 5 it follows that

$$\begin{aligned}&v^{(b)i}(x)-v^{(a)i}(x)+v^{(ab)i}(x)\\&\quad=\mp\Big[\pm\sum_\alpha e_\alpha \oiint_{\sigma_\alpha}\mathrm{d}s'^i\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big]\\&\quad=\mp\Big[\pm\partial^i\sum_\alpha e_\alpha\iiint_{V_\alpha}\mathrm{d}V'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big]\end{aligned} \tag{59a}$$

and

$$\begin{aligned}&v^{(b)0}(x)-v^{(a)0}(x)+v^{(ab)0}(x)\\&\quad=\mp\Big[\pm\sum_\alpha\frac{e_\alpha}{c}\oiint_{\sigma_\alpha}\mathrm{d}\boldsymbol{s}'\cdot\dot{\boldsymbol{r}}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big]\\&\quad=\mp\Big[\pm\partial^0\sum_\alpha e_\alpha\iiint_{V_\alpha}\mathrm{d}V'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\Big].\end{aligned} \tag{59b}$$

Here the first sign is determined by whether case (i) (−) or (ii) (+) holds and the second by whether $\mathrm{d}t/\mathrm{d}\tau$ is greater (+) or less (−) than zero. We introduce for the volume V_α invariant parameters u_α, v_α and w_α such that

$$\mathrm{d}V' = \pm \frac{\partial \boldsymbol{r}'}{\partial u_\alpha} \cdot \frac{\partial \boldsymbol{r}'}{\partial v_\alpha} \times \frac{\partial \boldsymbol{r}'}{\partial w_\alpha}\, \mathrm{d}u_\alpha\, \mathrm{d}v_\alpha\, \mathrm{d}w_\alpha \tag{60}$$

where the plus sign applies in case (i) and the minus sign in case (ii). An observer O ascribes to the points of V_α the four-vectors $x'(u_\alpha, v_\alpha, w_\alpha, \tau)$ having the property that $\partial x'^0/\partial u_\alpha$, $\partial x'^0/\partial v_\alpha$ and $\partial x'^0/\partial w_\alpha$ are all identically zero. Because of this the pseudoscalar function H_{V_α} defined by

$$H_{V_\alpha}(x) = \epsilon^{\mu\nu\rho\sigma} \int_{-\infty}^{\infty} \int_{u_\alpha^{(1)}}^{u_\alpha^{(2)}} \int_{v_\alpha^{(1)}}^{v_\alpha^{(2)}} \int_{w_\alpha^{(1)}}^{w_\alpha^{(2)}} \frac{\partial x'_\mu}{\partial u_\alpha} \frac{\partial x'_\nu}{\partial v_\alpha} \frac{\partial x'_\rho}{\partial w_\alpha} \frac{\partial x'_\sigma}{\partial \tau'} \delta(x - x')\, \mathrm{d}u_\alpha\, \mathrm{d}v_\alpha\, \mathrm{d}w_\alpha\, \mathrm{d}\tau' \tag{61}$$

can be written as

$$H_{V_\alpha}(x) = (\pm 1)(\pm 1) \iiint_{V_\alpha} \mathrm{d}V'\, \delta(\boldsymbol{r} - \boldsymbol{r}') = \begin{cases} 0, & \text{if } r \notin V_\alpha \\ (\pm 1)(\pm 1)(\pm 1), & \text{if } r \in V_\alpha. \end{cases} \tag{62}$$

In this last expression the first sign is determined by whether case (i) (+) or (ii) (−) holds, the second by whether $\mathrm{d}t/\mathrm{d}\tau$ is greater (+) or less (−) than zero and the third by whether the reference frame is right (+) or left (−) handed. It follows from equation (62) that equations (59) are of the form (58) with

$$\bar{\chi}^{(ab)}(x) = -\sum_\alpha e_\alpha H_{V_\alpha}(x). \tag{63}$$

4.3. *Transformation of pseudoscalar field $\bar{\chi}^{(ab)}$ with change of surface S^{ab}*

The transformation (58) connecting two given fields $v^{(a)}$ and $v^{(b)}$ can be carried out with various surfaces S^{ab} bounded by C^a and C^b. We consider two such surfaces S_A^{ab} and S_B^{ab} and their associated fields $v_A^{(ab)}$ and $v_B^{(ab)}$ defined as in equation (55). The fields $\bar{\chi}_A^{(ab)}$ and $\bar{\chi}_B^{(ab)}$ to be used in the transformation must then be related by

$$\partial^\mu (\bar{\chi}_B - \bar{\chi}_A) = v_B^\mu - v_A^\mu \tag{64}$$

where the superscript ab is now suppressed. We let V_{AB} be the volume bounded by S_A and S_B, which we assume to form a simple closed surface. We then have two cases according as (A) the normal to S_A or (B) the normal to S_B points into V_{AB} in a right handed frame and out of V_{AB} in a left handed frame. (Cases in which neither (A) nor (B) holds must be treated separately.) By an argument similar to the one leading to equations (59) we obtain

$$v_B^\mu - v_A^\mu = (\pm 1)(\pm 1)\partial^\mu Q \iiint_{V_{AB}} \mathrm{d}V'\, \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{65}$$

where the first sign is determined by whether case (A) (+) or case (B) (−) holds and the second by whether $\mathrm{d}t/\mathrm{d}\tau$ is greater (+) or less (−) than zero. We use for the volume V_{AB} invariant parameters u, v and w such that

$$\mathrm{d}V' = \pm \frac{\partial \boldsymbol{r}'}{\partial u} \cdot \frac{\partial \boldsymbol{r}'}{\partial v} \times \frac{\partial \boldsymbol{r}'}{\partial w}\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w \tag{66}$$

with the plus sign applying in case (A) and the minus sign in case (B). Then

$$H_{V_{AB}}(x) \equiv \epsilon^{\mu\nu\rho\sigma} \int_{-\infty}^{\infty} \int_{u^{(1)}}^{u^{(2)}} \int_{v^{(1)}}^{v^{(2)}} \int_{w^{(1)}}^{w^{(2)}} \frac{\partial x'_\mu}{\partial u} \frac{\partial x'_\nu}{\partial v} \frac{\partial x'_\rho}{\partial w} \frac{\partial x'_\sigma}{\partial \tau'} \delta(x-x')\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w\, \mathrm{d}\tau'$$

$$= (\pm 1)(\pm 1) \iiint_{V_{AB}} \mathrm{d}V'\, \delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$= \begin{cases} 0, & \text{if } \boldsymbol{r} \notin V_{AB} \\ (\pm 1)(\pm 1)(\pm 1), & \text{if } \boldsymbol{r} \in V_{AB} \end{cases} \tag{67}$$

where the first sign is determined by whether case (A) (+) or case (B) (−) holds, the second by whether $\mathrm{d}t/\mathrm{d}\tau$ is greater (+) or less (−) than zero and the third by whether the reference frame is right (+) or left (−) handed. It follows from equations (64), (65) and (67) that

$$\bar{\chi}_B(x) = \bar{\chi}_A(x) + QH_{V_{AB}}(x) + \kappa, \tag{68}$$

the pseudoscalar κ being a constant in any frame ($\partial^\mu \kappa = 0$).

It can readily be shown from the expressions (62) and (67) that for either of the cases ((A) or (B)) considered in this subsection taken in conjunction with either of the cases ((i) or (ii)) considered in the previous subsection, the relation

$$H_{V_\alpha^B}(x) = H_{V_\alpha^A}(x) - H_{V_{AB}}(x) \tag{69}$$

is valid. Here V_α^A is the volume bounded by Σ_α^a, Σ_α^b and S_A^{ab} and V_α^B is that bounded by Σ_α^a, Σ_α^b and S_B^{ab}. If $\bar{\chi}_A$ and $\bar{\chi}_B$ are defined as in equation (63), then we obtain from the relation (69)

$$\bar{\chi}_B(x) = \bar{\chi}_A(x) + QH_{V_{AB}}(x). \tag{70}$$

This is a particular instance, in which $\kappa = 0$, of the transformation (68).

5. Discussion

We have shown that the microscopic charge and current densities due to an aggregate of charged point particles are derivable in a covariant manner from polarisation–magnetisation tensor fields. Covariant expressions for these and other auxiliary fields have been found. Transformations of the fields have been investigated both in the general case and in the special case of line integral polarisation–magnetisation tensors. For a fixed orthochronous right handed reference frame, the formalism developed here reduces to that given in I. It should be noted that the behaviour of the fields under the time reversal and space inversion transformations was not considered in I. (It should also be noted that the symbol $\boldsymbol{m}$ which was used in I to denote the relative magnetisation field has been used in this paper to denote the total magnetisation field.)

In § 3.2 we proved that the line integral polarisation–magnetisation tensors have the same form for all observers related by homogeneous Lorentz transformations. The argument that was used depended on the speed of every point of the integration paths being less than c. A similar argument can be used to establish an analogous property for the other fields that appear in the theory. The motion of all the curves, surfaces and volumes must then be restricted so that the speed of every point of them is always less than c. We note that this motion has not been dynamically linked to that of the material particles, except insofar as some of the curves must end at the particle positions. Both this paper and paper I have thus been concerned with a purely kinematical description of polarisation and magnetisation fields.

Appendix. Theorems in vector analysis

Let f and $\boldsymbol{F}$ be three-dimensional scalar and vector fields. Let C be a closed and sensed curve bounding a surface S and Σ a closed and sensed surface bounding a volume V. Both the surface S and the volume V may be moving and f and $\boldsymbol{F}$ may depend explicitly on the time t and the fixed field point $\boldsymbol{r}$ as well as on the moving, and thus time dependent, source point $\boldsymbol{r}'$. The sense of C is taken to be independent of the handedness of the coordinate frame. The sense of S is determined from that of C by the right-hand rule in a right handed frame and the left-hand rule in a left handed frame. The normal to Σ will be taken to point outwards in a right handed frame and inwards in a left handed frame. Any volume element of V will be reckoned positive in a right handed frame and negative in a left handed frame. Then the following theorems are true:

Theorem 1

$$\oint_C f\,\mathrm{d}\boldsymbol{r}' = \iint_S \mathrm{d}\boldsymbol{s}' \times \nabla' f. \tag{71}$$

Theorem 2

$$\oint_C \mathrm{d}\boldsymbol{r}' \times \boldsymbol{F} = \iint_S (\mathrm{d}\boldsymbol{s}' \times \nabla') \times \boldsymbol{F}. \tag{72}$$

Theorem 3

$$\oiint_\Sigma f\,\mathrm{d}\boldsymbol{s}' = \iiint \nabla' f\,\mathrm{d}V'. \tag{73}$$

Theorem 4

$$\frac{\partial}{\partial t}\iint_S \mathrm{d}\boldsymbol{s}'\,\delta(\boldsymbol{r}-\boldsymbol{r}') + \nabla \iint_S \mathrm{d}\boldsymbol{s}' \cdot \dot{\boldsymbol{r}}'\,\delta(\boldsymbol{r}-\boldsymbol{r}') = -\iint_S (\mathrm{d}\boldsymbol{s}' \times \nabla') \times \{\dot{\boldsymbol{r}}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\}. \tag{74}$$

Theorem 5

$$\iiint_V \mathrm{d}V' \nabla' \cdot \{\dot{\boldsymbol{r}}'\,\delta(\boldsymbol{r}-\boldsymbol{r}')\} = \frac{\partial}{\partial t}\iiint_V \mathrm{d}V'\,\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{75}$$

The proof, or the method of proof, of each of these theorems (as well as of one other theorem not used in this paper) was outlined in I where the analysis was implicitly restricted to right handed reference frames. But the proofs hold equally well for left handed frames, provided the senses of S and Σ and the sign of $\mathrm{d}V'$ are chosen in the way described above. It may also be noted that theorems 4 and 5, which involve time derivatives, are covariant under the time reversal transformation.

References

de Groot S R 1969 *The Maxwell Equations* (Amsterdam: North-Holland)
de Groot S R and Suttorp L G 1965 *Physica* **31** 1713
de Groot S R and Vlieger J 1965 *Physica* **31** 125
Healy W P 1977 *Proc. R. Soc. Lond.* A **358** 367